7. Three-Sided Dice
4. Unsinkable Disk

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7. Three-Sided Dice

To land a coin on its side is often associated with the idea of a rare occurrence. What should be the physical and geometrical characteristics of a cylindrical dice so that it has the same probability to land on its side and one of its faces?

History

Rebounding Capsule (34th IYPT)

A spherical ball dropped onto a hard surface will never rebound to the release height, even if it has an initial spin. A capsule-shaped object (i.e. Tic Tac mint) on the other hand may exceed the initial height. Investigate this phenomenon.

Probability (19th IYPT)

A coin is held above a horizontal surface. What initial conditions will ensure equal probability of heads and tails when the coin is dropped and has come to rest?

Coin (10th IYPT)

From what height must a coin with heads up be dropped, so that the probability of landing with heads or tails up is equal?

Simple cases

Thin disk | **Thin rod**

Probability of landing on any face

Probability of landing on any face

Background photo by Pixabay.com

Simple cases

Thin disk | **Thin rod**

Probability of landing on any face
100%

Probability of landing on any face
0%

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Simple cases

Thin disk	Thin rod

Probability of landing on any face	Probability of landing on any face
100%	**0%**

Really always?

Background photo by Pixabay.com

Simple cases

Thin disk | **Thin rod**

Probability of landing on any face

Probability of landing on any face

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Simple cases

Thin disk

Thin rod

Probability of landing on any face

Probability of landing on any face

Background photo by Mudd1

Simple cases

Thin disk | **Thin rod**

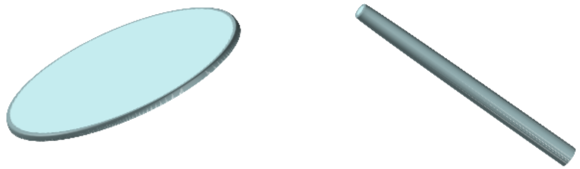
Probability of landing on any face **< 100%**

Probability of landing on any face **> 0%**

Background photo by Mudd1

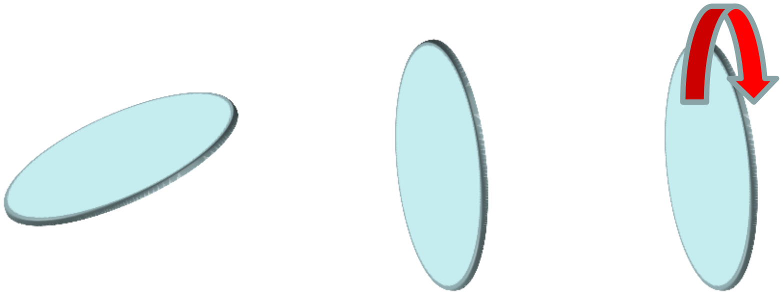
Important parameters

- Geometry ("the dice")



- Target surface ("where the dice falls")

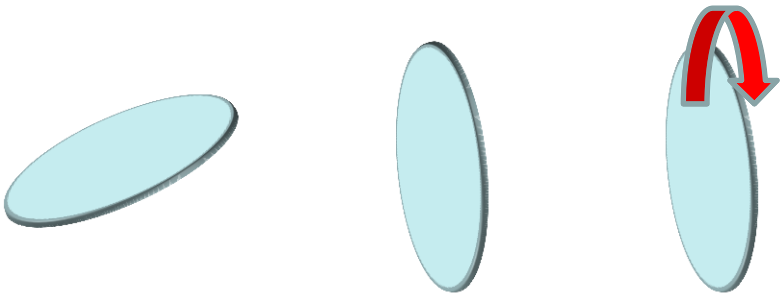
- Initial conditions ("how the dice is thrown")



To land a coin on its side is often associated with the idea of a rare occurrence. What should be the physical and geometrical **characteristics of a cylindrical dice** so that it has the same probability to land on its side and one of its faces?

Important parameters – initial conditions

- Initial conditions for rigid body motion → 12 parameters
  - Position
  - Rotation
  - Velocity
  - Rotation speed



  - Reduction due to symmetry
    - Symmetrical cylinder in free space → 6 parameters "only" (height, lateral velocity and fall velocity, tilt, tilt speed and rotation speed along the axis)
    - Cylinder with offset center of gravity → 8 parameters

Important parameters – target surface

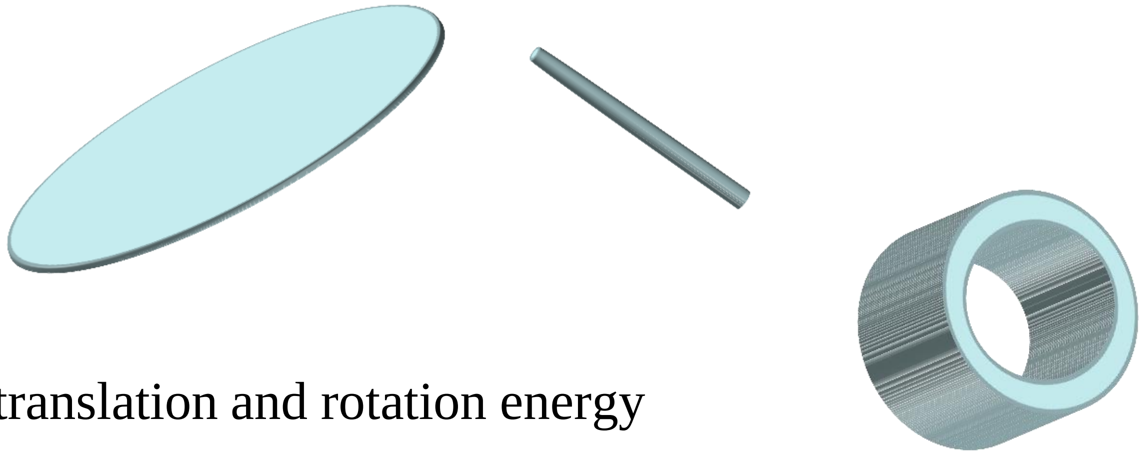
- Target surface, dice-surface interaction
  - Coefficient of restitution
  - Coefficient of friction
  - Shape of the surface

- Simplifications
  - No friction/Infinite friction
  - Coefficient of restitution $\rightarrow 0$ (e.g. landing in flour)

- Literature: W. Goldsmith, *Impact*. London, Arnold (1960)

Important parameters – geometry

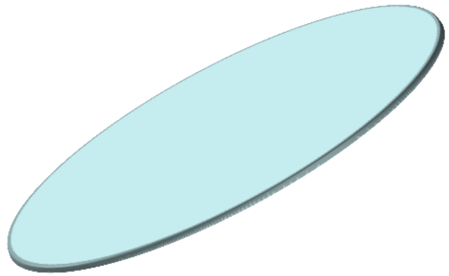
- Aspect ratio
  - Controls the stability

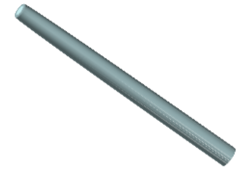


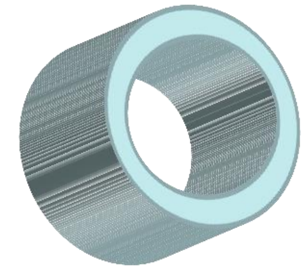
- Mass distribution
  - Controls the ratio of translation and rotation energy

Important parameters – geometry

- Aspect ratio
  - Controls the stability





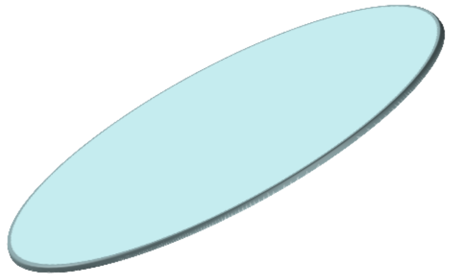


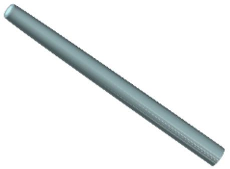
- Mass distribution
  - Controls the ratio of translation and rotation energy

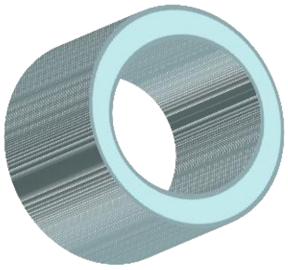
When are important the following parameters?

- Size
- Density of material
- (Airflow)

Important parameters – geometry







- Aspect ratio
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- Mass distribution
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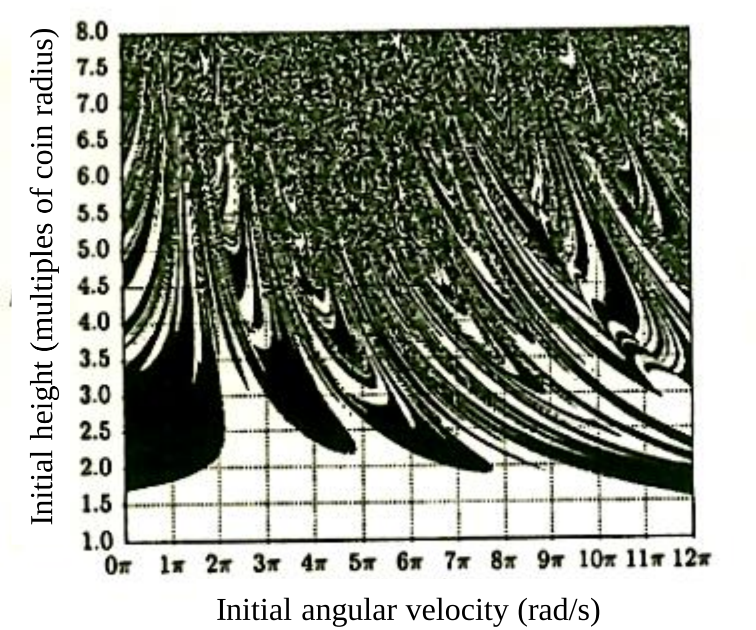
$\left.\begin{array}{l}\text{Size}\\ \text{Density of material}\end{array}\right\}$ Force $\not\propto$ mass

Large speeds and/or heights

Deterministic approach

- Initial conditions + dice geometry and physical parameters
- Dynamics of rigid body (gyroscope equation)
  - 1D: Newton’s equations of translation and rotation motion
- Impact to the surface



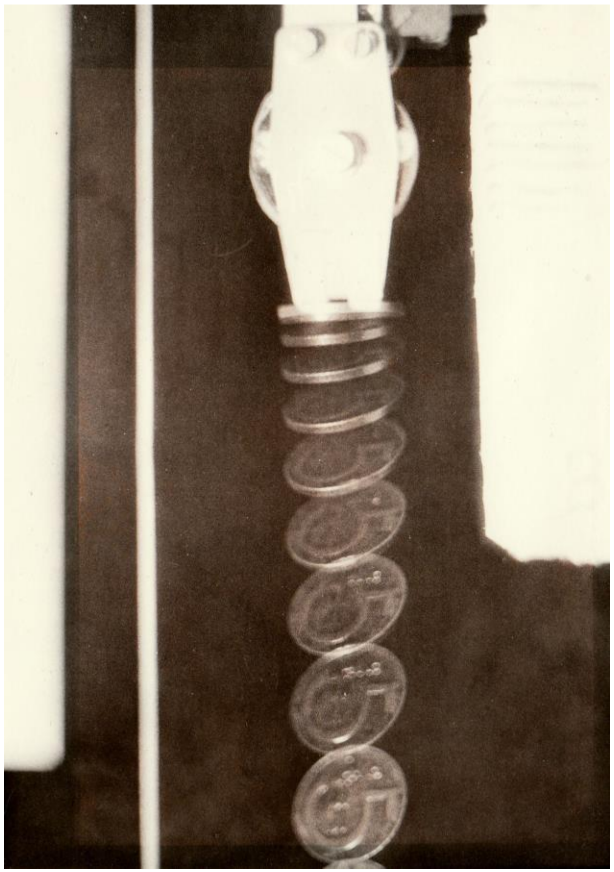
Example: calculated diagram showing the terminal state of a 5 CZK coin on a classroom table.

White = same orientation

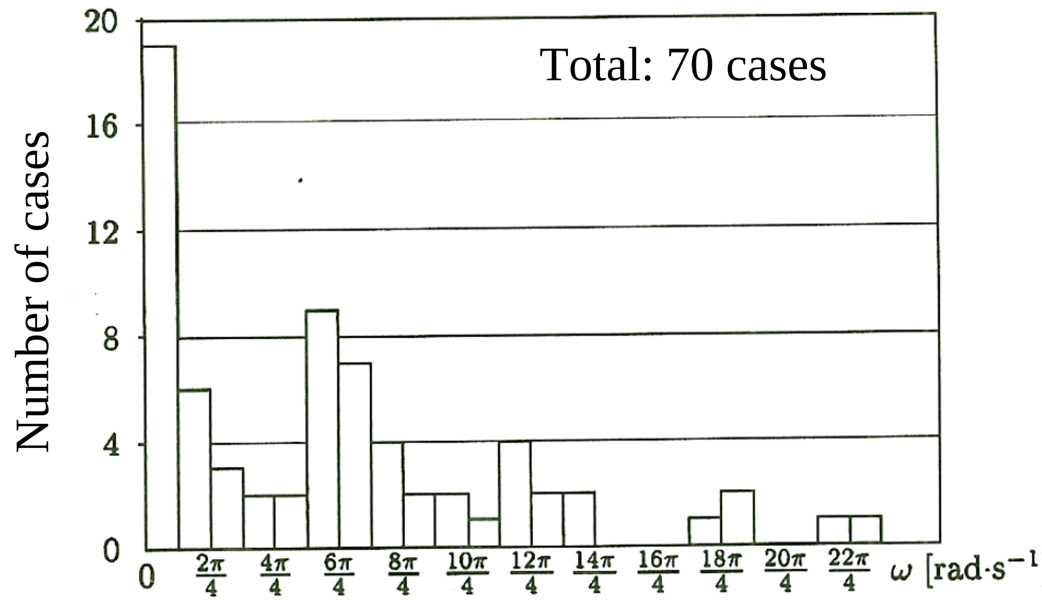
Black = opposite orientation

[In: Nápaditá fyzika, 1. vydání, ARSCII, Praha (2000), pp. 53 – 62]

- Issue: Control/characterization of initial conditions
  - Typically a distribution



Motion of the 5 CZK dropped by an electromagnet



Initial angular velocity of the tilt (rad/s)

In: Nápaditá fyzika, 1. vydání, ARSCII, Praha (2000), pp. 53 – 62

Stochastic approach

- Suitable distribution of initial conditions, e.g.
  - Initial height around 1.20 m
  - Shaking of dice assuring (quasi-)random distribution of initial tilts and a distribution of velocities

- The dice has some kinetic energy before the impact. Assess whether it may change the state upon collision
- Examples
  - Kinetic energy in the form of rotation along the main axis
  - Kinetic energy in the form of tilting
- Vary the dice shape to test your hypothesis

Statistics

If the probability p of an event (e.g. cylinder landing face up) remains constant, then the probability $P_n(k)$ that the event occurs exactly k-times out of n cases follows **binomial distribution**

$$P_n(k) = \binom{n}{k} p^k (1-p)^{n-k} \equiv \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k}$$

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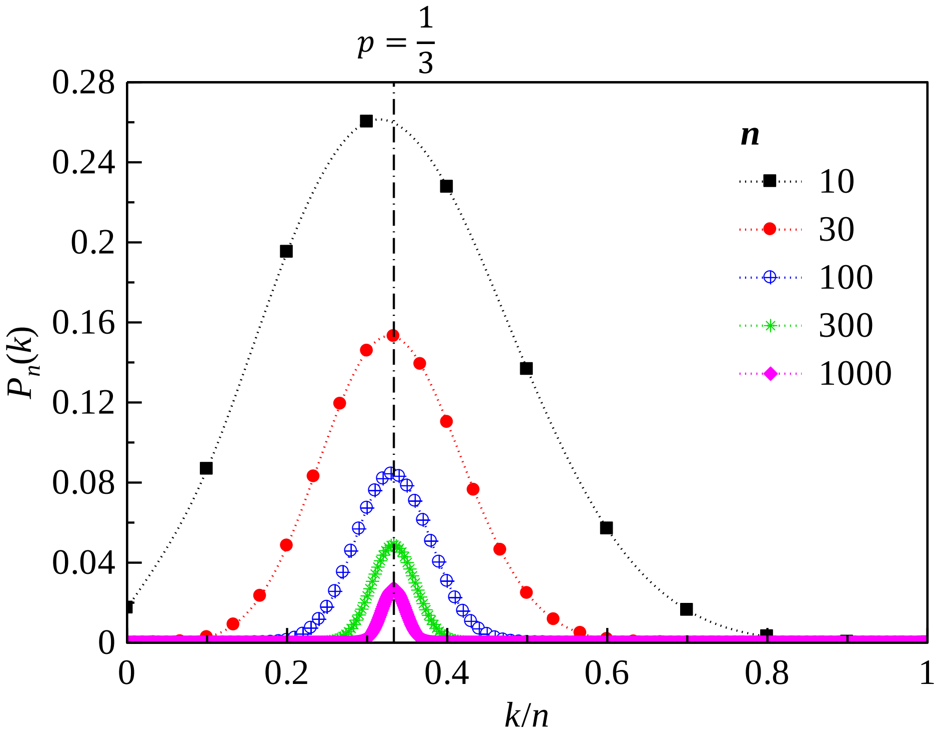

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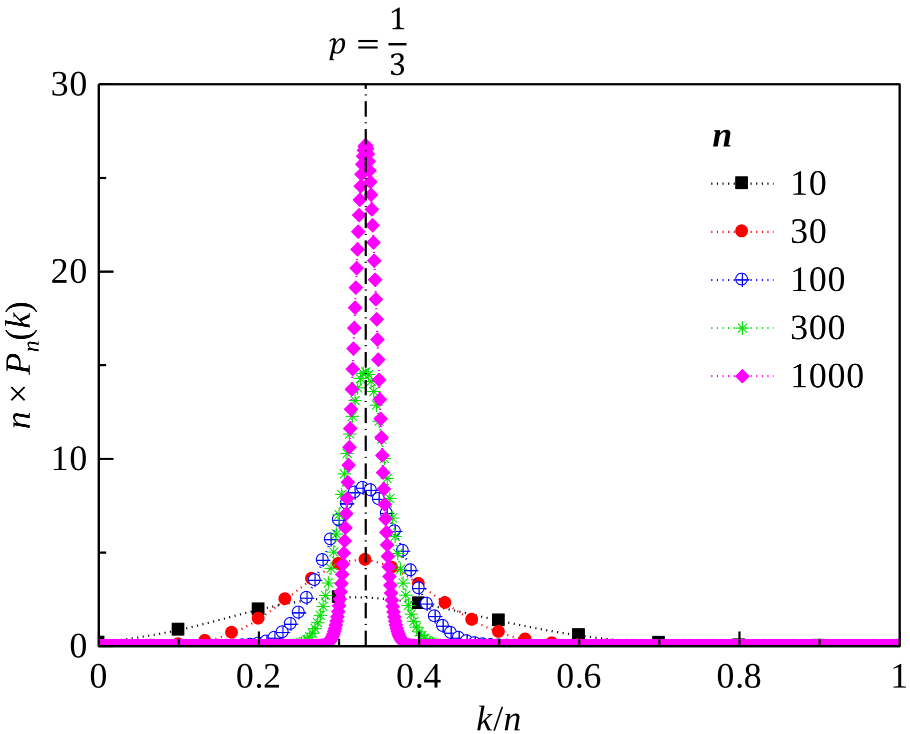

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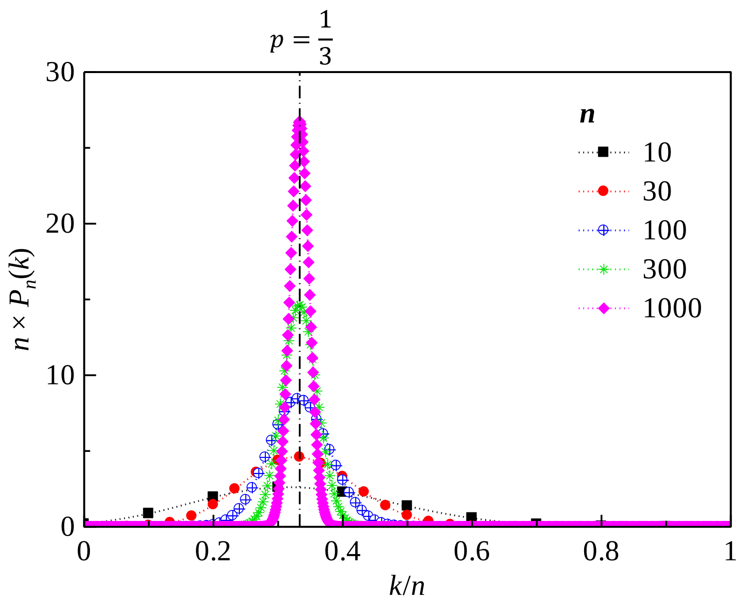

Important characteristics

$$\frac{\text{Mean value}}{n} = p$$

$$\frac{\text{Standard deviation}}{n} = \sqrt{\frac{p(1-p)}{n}}$$

Statistics

The phenomenon happens with probability p.

What frequency shall we measure?

$n \setminus p$	1%	3%	10%	33.3%
10	1 ± 3	3 ± 6	10 ± 9	33 ± 15
30	1.0 ± 1.8	3 ± 3	10 ± 5	33 ± 9
100	1.0 ± 1.0	3.0 ± 1.8	10 ± 3	33 ± 5
300	1.0 ± 0.6	3.0 ± 1.0	10.0 ± 1.7	33.3 ± 3
1 000	1.0 ± 0.3	3.0 ± 0.6	10.0 ± 0.9	33.3 ± 1.5
3 000	1.00 ± 0.18	3.0 ± 0.3	10.0 ± 0.5	33.3 ± 0.9
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Fundamental questions

- How many times should we throw the dice?
- How long this will take?
- When shall we become tired?

Statistics

The phenomenon happens with probability p.

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However (σ denotes the standard deviation)

The result fits within ±σ with 68% probability

The result fits within ±2σ with 95% probability

The result fits within ±3σ with 99.7% probability

Statistics

The phenomenon happens with probability p.

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Balance between accuracy and time: $\boxed{\text{Accuracy} \propto \sqrt{\text{time}}}$

How to start

- Calculate the probability of the results for a dice in contact with the surface (consider that the orientations are random) as a function of the aspect ratio

- Perform many experiments under selected conditions. Observe whether the results agree with the simple picture:
  - Yes? ⇒ Explain why the complex dynamics leads to so simple observation!
  - No? ⇒ Explain the difference: any deviation from the simple view is interesting!

Win-win situation!

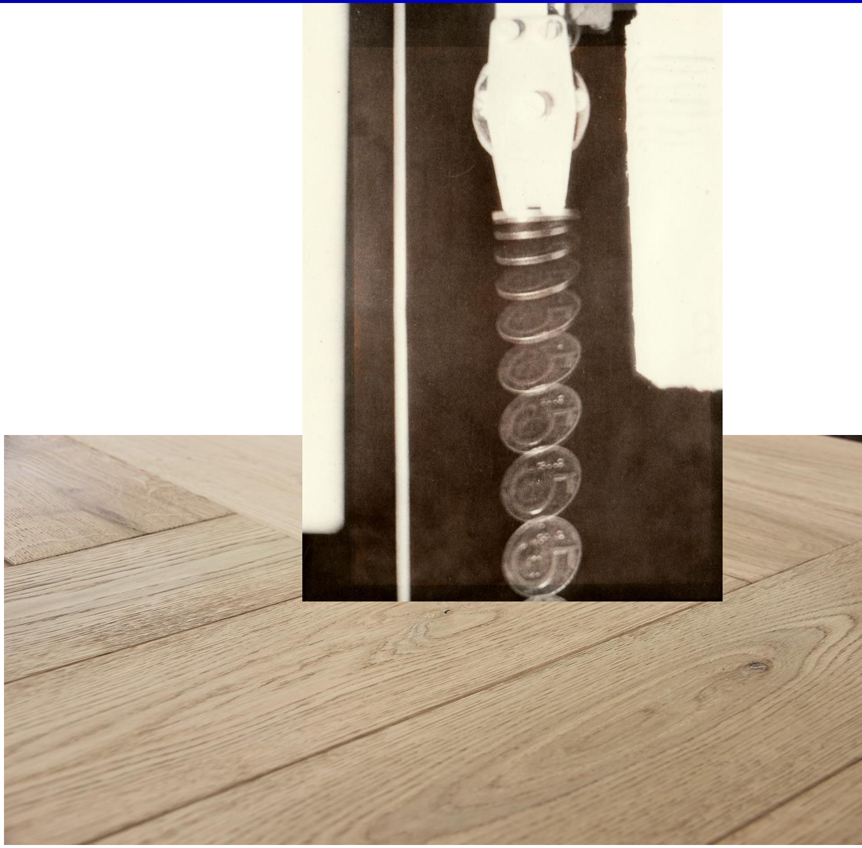
- Select a parameter which you vary. Ensure that the other remain as intact as possible!

Conclusions (Three-Sided Dice)

- Important parameters
  - Dice
    - Aspect ratio, mass distribution
  - Dice-surface interaction
    - Coefficient of restitution and friction
  - Surface
    - Shape
  - Initial conditions
    - Random or deterministic

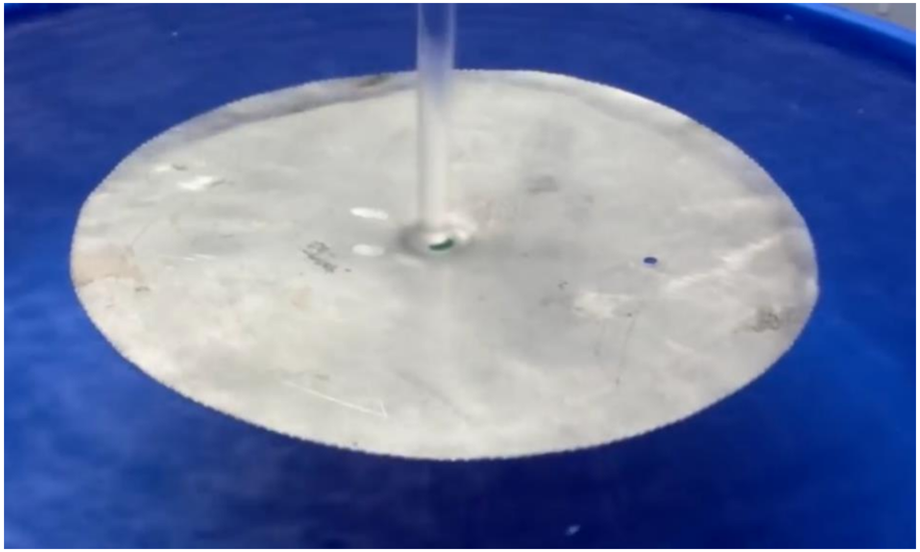
- Approaches
  - Deterministic or stochastic

- Statistics
  - Doing < 100 realizations makes little sense



4. Unsinkable Disk

A metal disk with a hole at its centre sinks in a container filled with water. When a vertical water jet hits the centre of the disc, it may float on the water surface. Explain this phenomenon and investigate the relevant parameters.



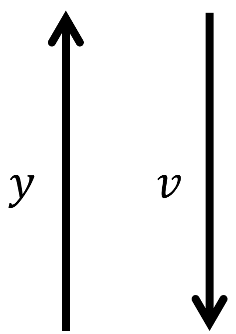
https://www.youtube.com/watch?v=eP5_9eUjfkI

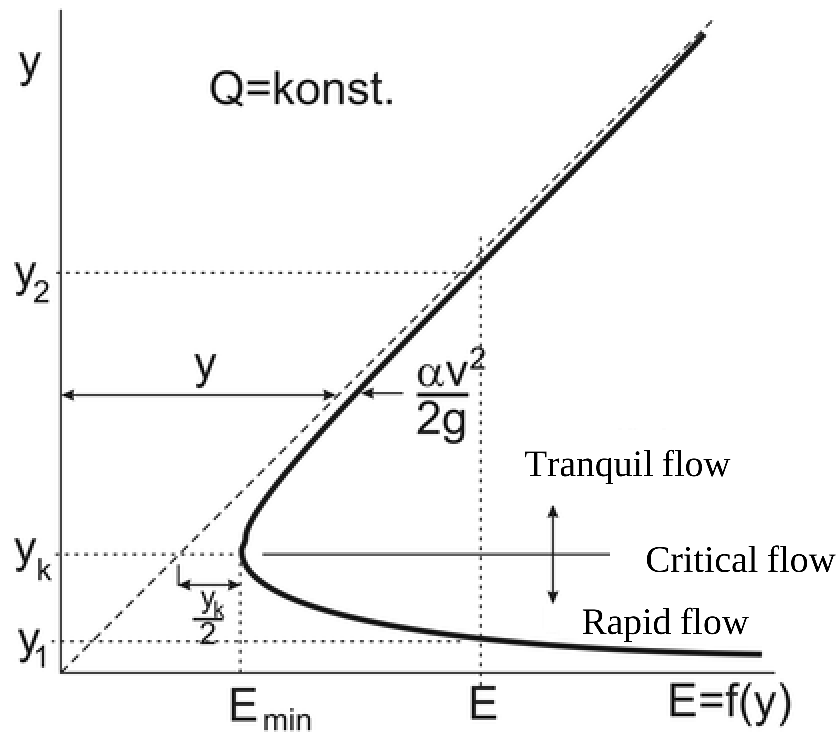
Open-channel flow

Energy of a flow with velocity v in a open channel with depth y

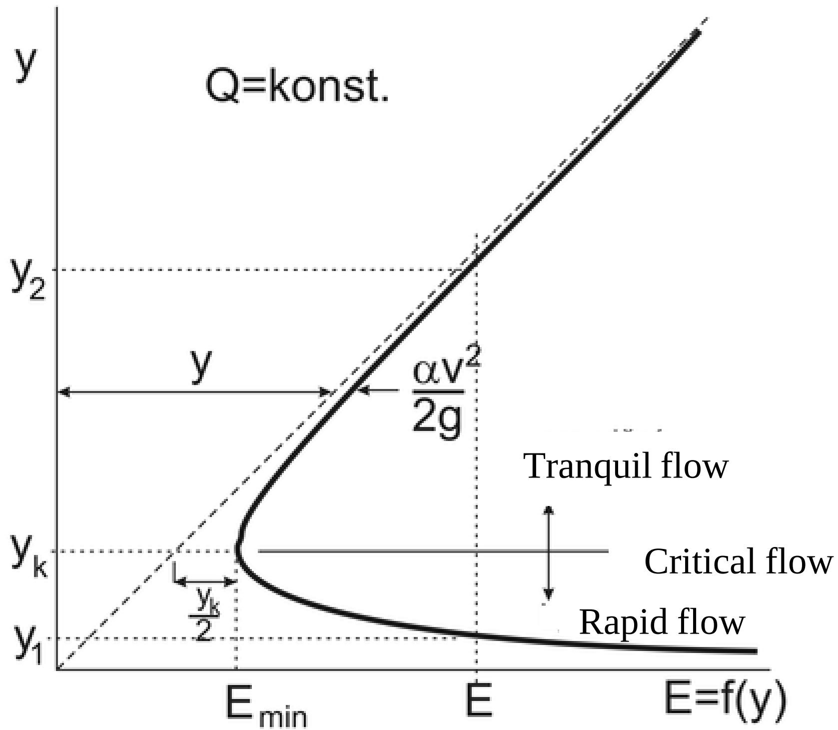
$$\frac{E}{\varrho g} = y + \frac{v^2}{2g}$$

Let's assume a constant flow rate Q ($\propto v \cdot y \Longleftarrow$ mass conservation)





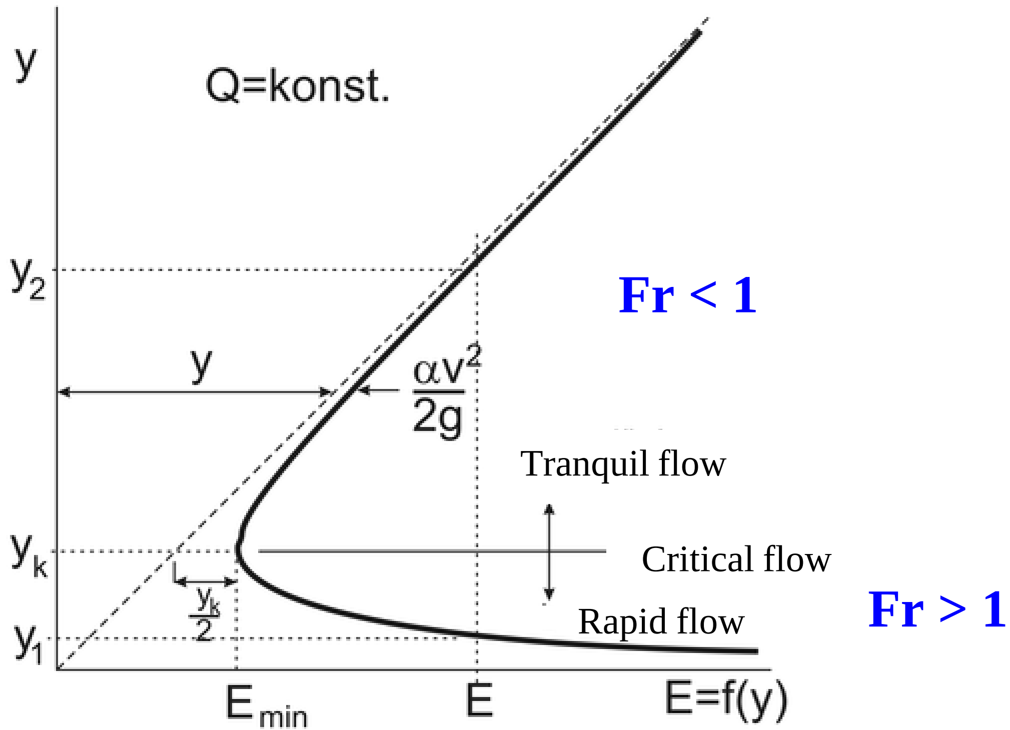
Energy of a flow



Critical depth y_k: transition between tranquil and rapid flow

Open-channel flow

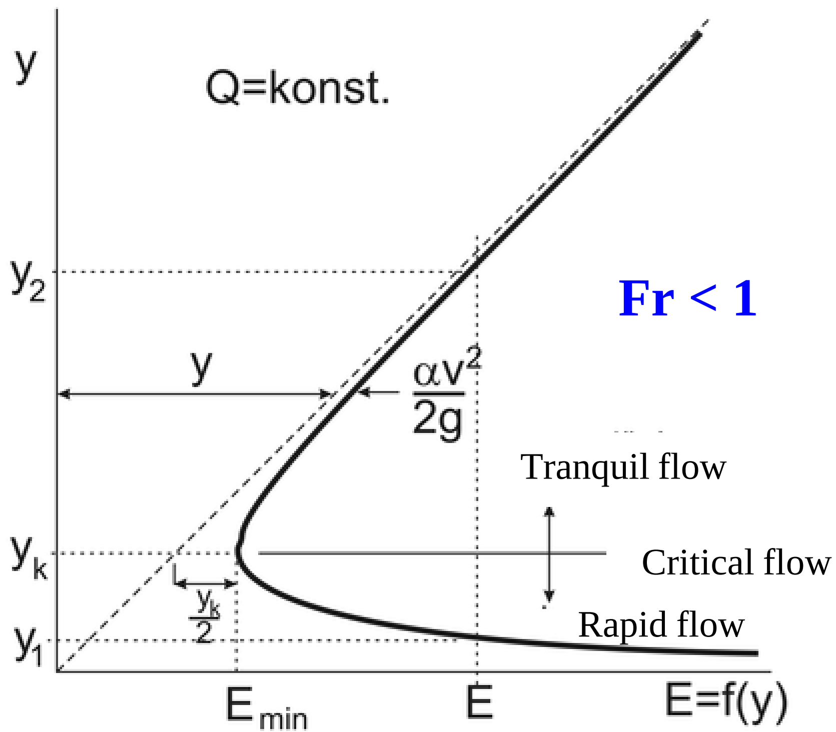
Energy of a flow



Froude number (dimensionless number in fluid dynamics)

$$\mathrm{Fr} = \frac{v}{\sqrt{gy}} = \frac{\text{flow inertia}}{\text{gravity}}$$

Tranquil-rapid flow transition → instability → hydraulic jump

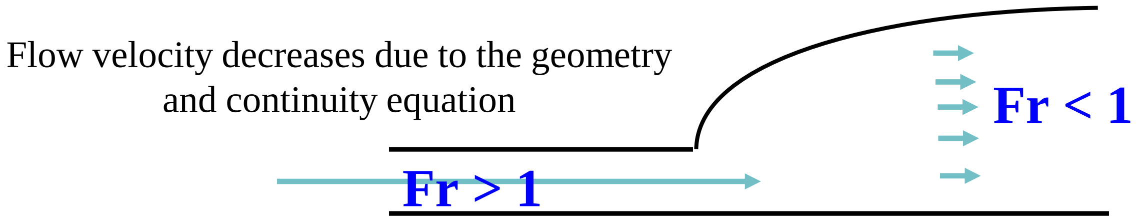


Fr < 1

Fr > 1

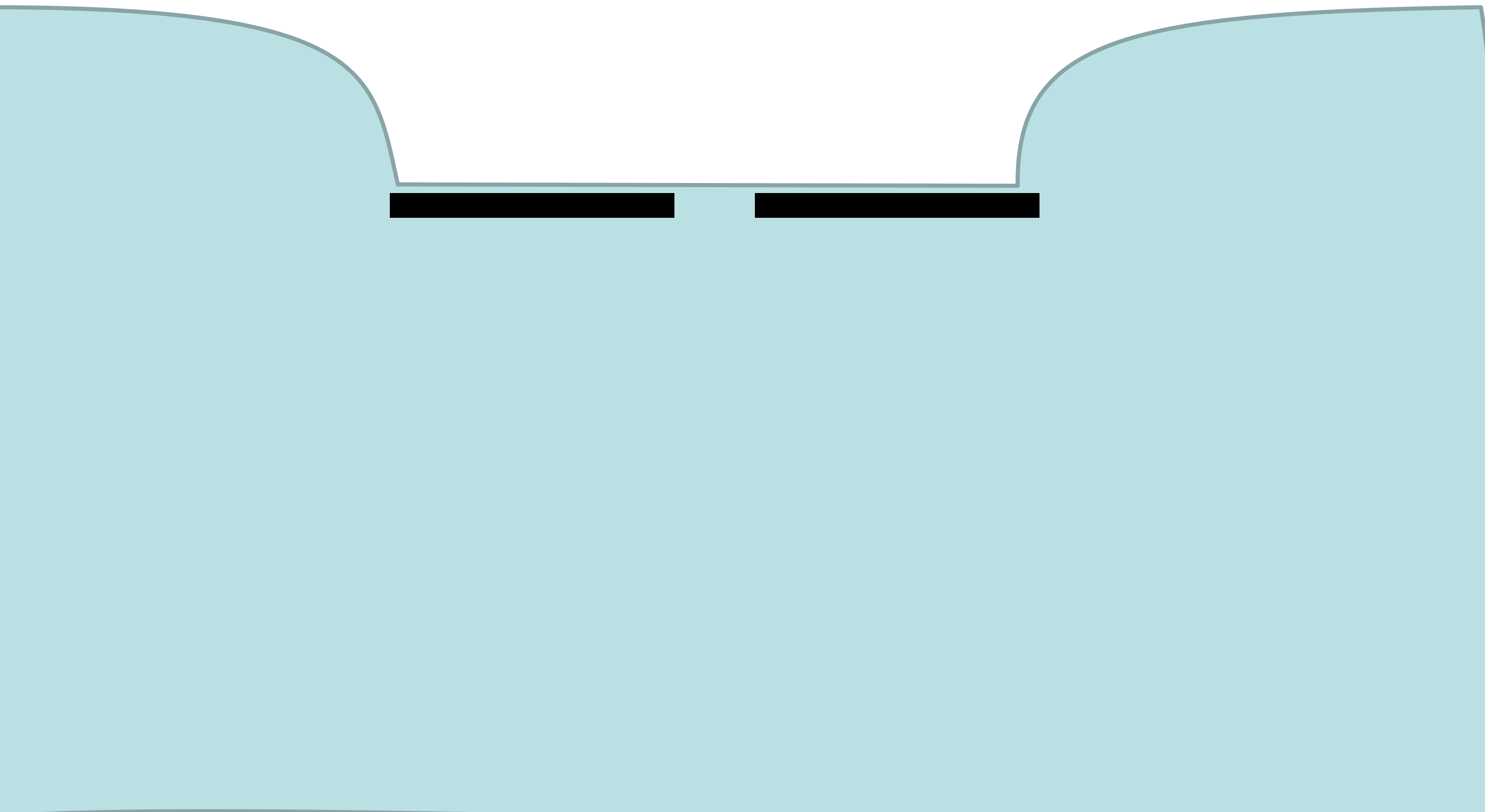
Hydraulic jump

Tranquil-rapid flow transition → instability → hydraulic jump



Buoyancy

Archimedes law



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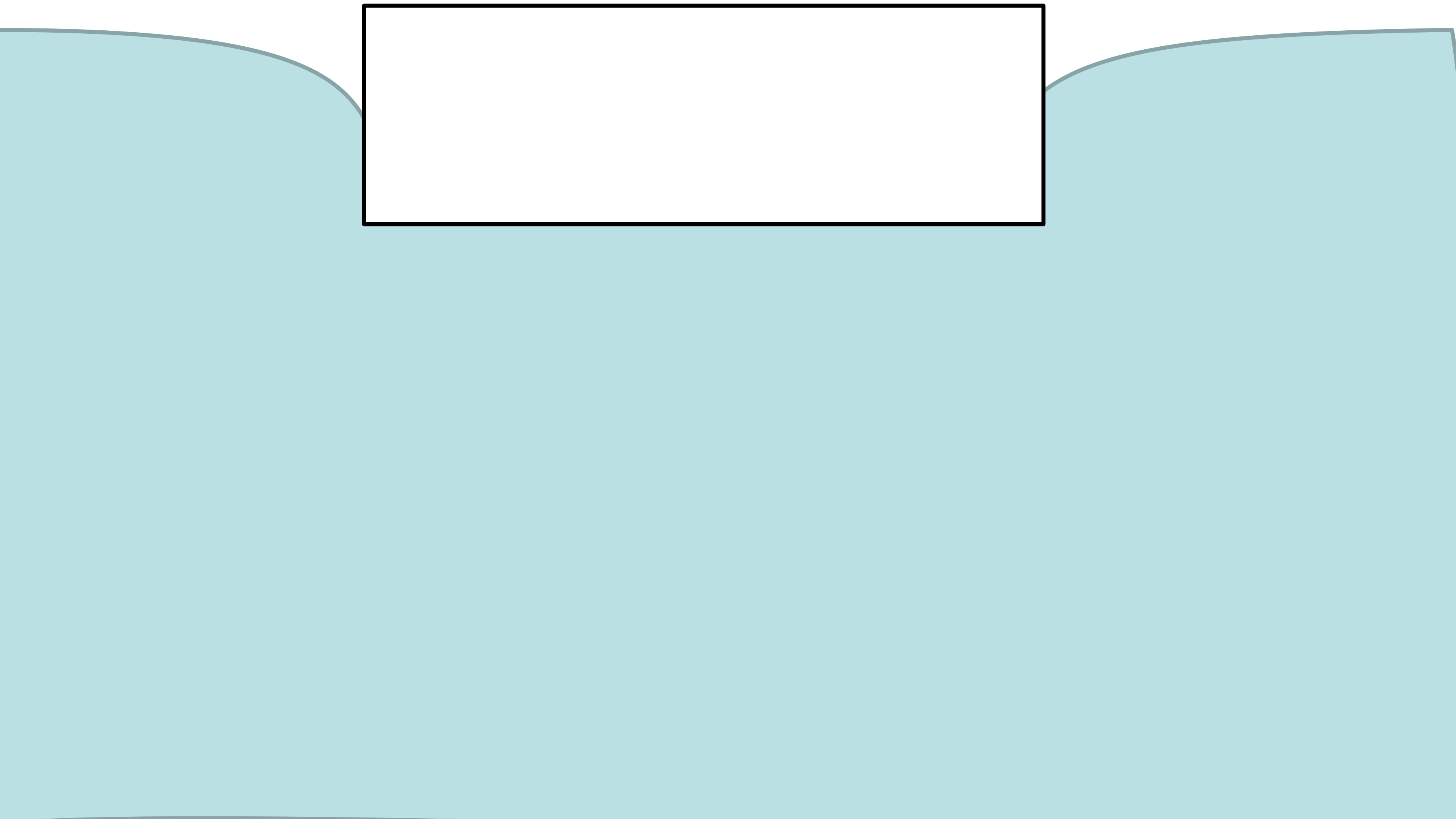
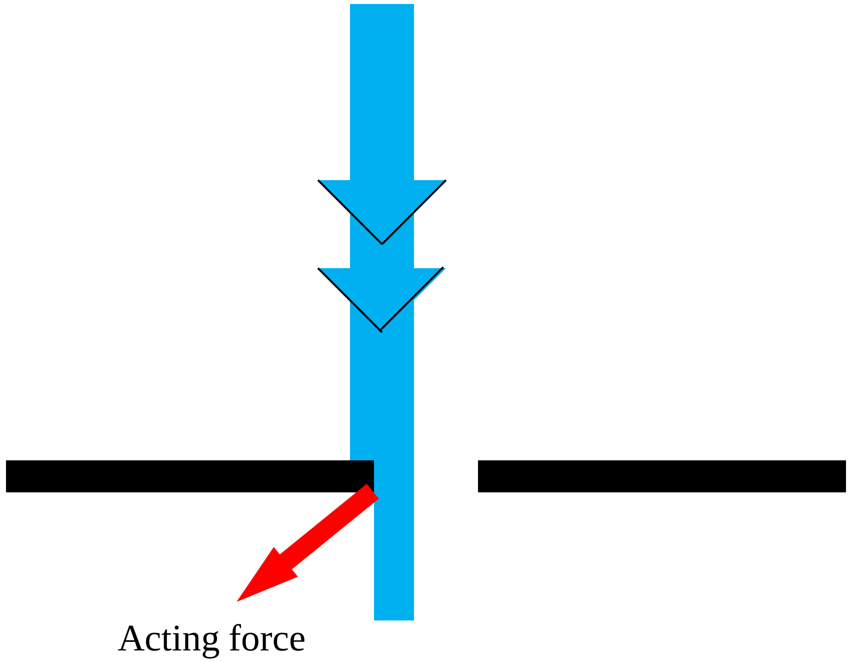


Stabilization

Restoring force



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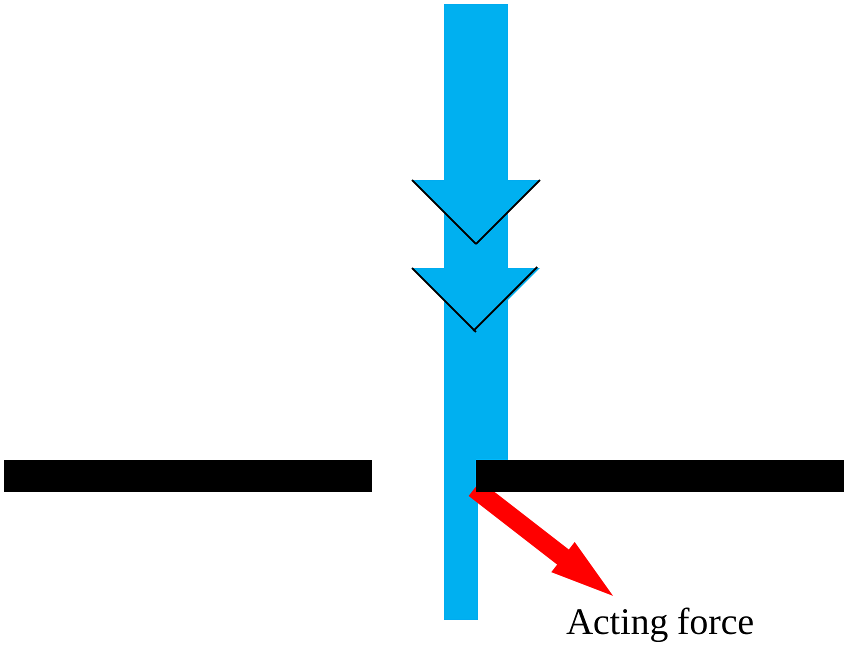
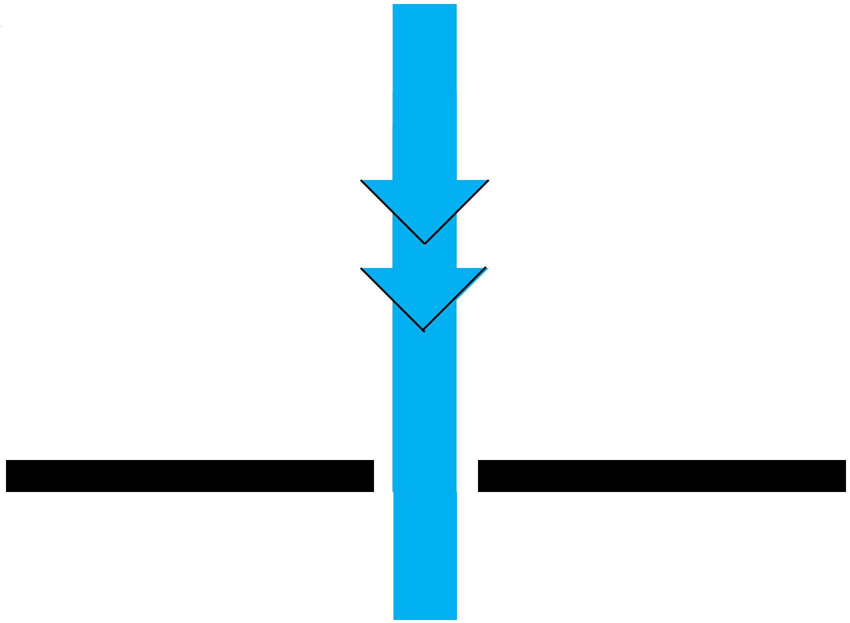


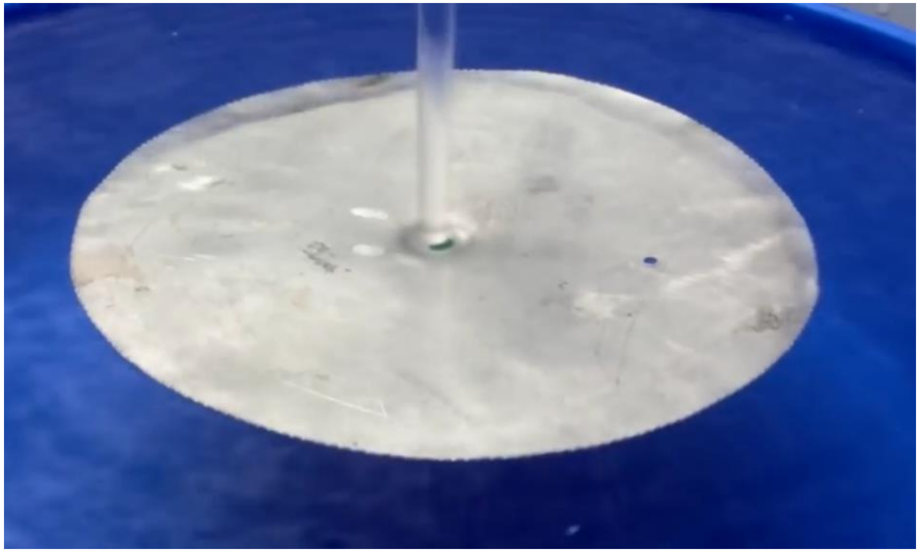
Stabilization

Optimal position



Conclusions (Unsinkable Disk)

- Hydraulic jump
  - Expels water from the space above the disc
- Buoyancy force
  - Pushes the disc up
- Restoring force
  - Prevents the disc from escaping



https://www.youtube.com/watch?v=eP5_9eUjfkI

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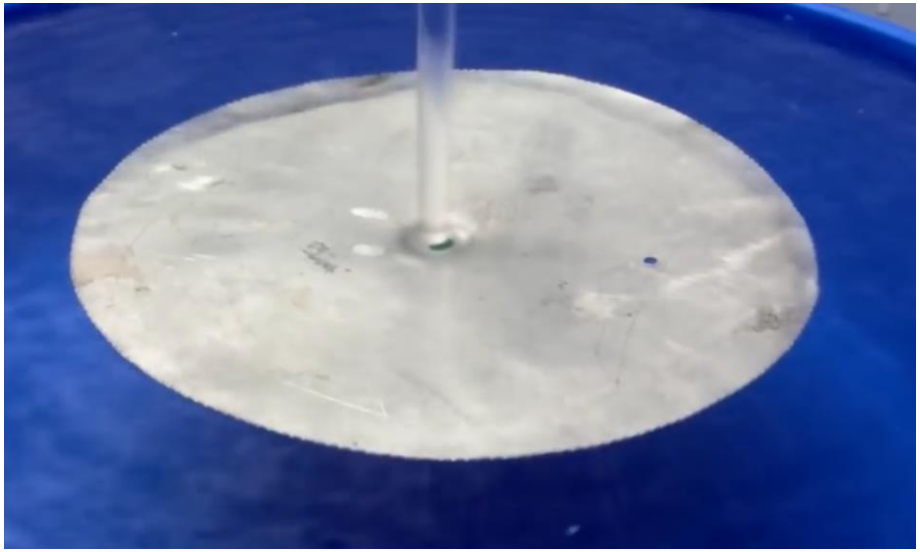
Plenty of effects worth investigation



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